

ON MATCHING BOOK: A PROBLEM IN BANKING AND CORPORATE FINANCE*

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In each of the asset and liability markets in which the banking firm is an intermediary, typically there are instruments with differing maturities. The bank matching book problem is to manage the term structures of assets and liabilities. In our first model, the bank borrows and lends only short run. In our second model, the bank borrows only short run but can lend short run and long run. The criterion in both models is the expected value of the present value of dividends issued. Both models yield dynamic programming problems. Broad aspects of optimal policies are indicated. (BANKING; MATCHING BOOK; DYNAMIC MODEL; INTEREST RATES; INVENTORIES)

1. Introduction

A firm may face a fluctuating demand for product produced from long-lived equipment. The equipment may need long-term financing although the fluctuation in income is short term. Most ordinary individuals can lend at a lower rate of interest than the rate at which they can borrow; this is also true of many firms. There are some large firms whose financial power is sufficiently large that the reverse may be true. They are implicitly, or even openly in part, finance companies. Banks in general are in a position to lend at rates higher than those at which they borrow.

Both banks and firms strive to match their short-term and long-term financial flows, but a bank's problem is somewhat simpler. In general the production processes of the firm are far more complex than those of a bank. In the bank matching book problem, the bank tries to optimize the structure of its long-run and short-run borrowing and lending. Although our primary interest is in the finance of the firm, the bank's structure is a natural preliminary to the more difficult corporate finance problem. The criterion in our model is maximization of the expected value of the sum of the discounted dividends issued by the bank.

The banking firm has a matching problem due to its role as an intermediary between asset and liability markets. Most of the literature on the theory of the banking firm consists of static (one-period) models that optimize the bank's behavior in one of these markets or the other; see Baltensperger (1980) and Santomero (1984) for surveys.

The treatment of asset and liability management as independent problems can be justified in a particular static model [cf. Klein 1971], but the independence is not warranted in other static models or in dynamic models [Baltensperger 1980 and Santomero 1984]. Deshmukh, Greenbaum and Kanatas (1982, 1983) present static and two-period models of coordinated asset and liability management. In their static model they deduce the effects of increased interest rate uncertainty, and in their two-period models they study a deposit-funding environment and a liability management environment.

An early dynamic model in Daellenbach and Archer (1969) concerns optimal short-run cash balance adjustments via borrowing and lending. In the dynamic model in O'Hara (1983), the bank selects short-run levels of consumption, deposit services and loans; the dividend is an exogenous function of each period's state.

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“Liquidity risk” is the possibility that depositors may withdraw some or all of their funds, and “default risk” is the possibility that borrowers may not repay all their debts when due. It seems to us that default risk should be imbedded in a model that explicitly describes the effects of bankruptcy and the attitude to risk of the banking firm’s manager or owner. Our models include liquidity risk, exclude default risk, and are consistent with risk neutrality. Our results can be viewed as stepping stones to the analysis of more elaborate models with risk sensitivity, default risk, and bankruptcy provisions.

The paper has the following organization. §2 discusses the modeling of a bank’s borrowing and lending operations. §3 investigates a model in which the bank matches only its short-term borrowing and lending opportunities in order to maximize the expected present value of the time stream of its dividends. Each period, the bank examines the amount of its cash reserves (a) and observes the current interest rates to borrow (ρ) and lend (η) short-term funds. It decides how much of a dividend (p) to issue and how much to borrow (D); the exogenous demand for short-term loans (Q) is revealed, and the amount of the cash reserves is updated.

§3 shows that the bank’s problem corresponds to a dynamic program with the following properties. There is an unconstrained optimization problem with parameters ρ and η and two decision variables; say $[y(\rho, \eta), d(\rho, \eta)]$ is an optimal solution. Then it is optimal for borrowing and the dividend to be nil ($p = D = 0$) if the cash level is too low and a leverage constraint is not binding. If the cash level is high enough and a leverage constraint is binding then the optimal amount to borrow is $D = D(\rho, \eta)$ and the optimal dividend is $a - y(\rho, \eta) - \rho D(\rho, \eta)$. Moreover, if η rises, the dividend does not increase and the amount borrowed does not decrease. §3 ends with a numerical example. §4 extends the model in §3 by permitting the probability distribution of the interest rate at which the bank borrows funds to depend on the amount of the bank’s cash reserves.

The model in §5 includes opportunities for the bank to lend funds either short term (x) or long term (z) even though it can secure only short-term funds (D). As in the model in §3, the bank maximizes the expected present value of the time stream of its dividends. Short-term loans are repaid at the end of the same period in which they are borrowed; long-term loans are repaid at the end of the subsequent period. Each period, the bank observes interest rates for lending funds short term (η) and long term (β) and for borrowing funds short term (ρ). Also, it recalls two data from the previous period: the interest rate for lending funds long term (β') and the amount of funds lent long term (Z); let $w = (1 + \beta')Z$. Augmenting the model in §3 and §4 with an additional state variable and an additional decision variable dramatically complicates the analysis.

The dynamic program that corresponds to the bank’s problem in §5 is shown to have the following properties. There is an unconstrained optimization problem that depends on parameters w and ρ and two decision variables; say $[y(\rho, w), d(\rho, w)]$ is an optimal solution. If the available deposits are not too high, namely $D < d(\rho, w)$, and the level of liquid assets is high enough, namely $a \geq y(\rho, w)$, then it is optimal to accept all of the available deposits. If the level of liquid assets is too low, namely $a < y(\rho, w)$, then it is optimal neither to issue a dividend nor to accept any deposits. The amounts loaned short term and long term, respectively, are nondecreasing functions of the amounts demanded short term and long term.

§6 extends the results in §3, §4, and §5 to an infinitely long planning horizon. Other considerations, such as bankruptcy, are mentioned in §7.

2. Modeling Considerations

There are no canonical models of the decision problem faced by a bank which borrows from sources of funds (depositors, savings accounts, central bank or elsewhere) and lends to customers. Building even a simple model of such a bank entails numerous choices in

the modeling of process details. So the significance of our results depends on the quality of our choices. These choices are discussed in this section to motivate the modeling and to illustrate the distinctions in realism, relevance and tractability of the models.

A. *The Rate of Interest for Borrowing by the Bank*

(i) *The Competitive Assumption.* The bank can obtain money at a given rate ρ_t . Its borrowing at ρ_t could be subject to a bound on size.

(ii) *The Oligopolistic or Imperfect Competition Assumption.* The bank can obtain funds at the rate $\rho_t = f(s_t, D_t)$ where s_t is the level of assets held by the bank and D_t is its level of debts at time t . We make no assumptions about $f(\cdot, \cdot)$ in this paper although one might expect it to have certain properties. These include homogeneity of order one, i.e. doubling assets and debt should not influence the rate at which the bank borrows. If debt is increased for a fixed level of assets, the borrowing interest rate of the bank may increase, thus $\partial f / \partial D_t > 0$ and $\partial^2 f / \partial D_t^2 > 0$ and if assets increase for no extra debt the borrowing position is improved, thus $\partial f / \partial s_t < 0$. We might expect $\partial^2 f / \partial s_t^2 \geq 0$.

B. *The Rate of Interest for Lending by the Bank*

(i) *The Competitive Assumption.* The bank lends at a fixed rate η .

(ii) *The Imperfect Competition Assumption.* The bank can control the rate of interest at which it lends. The relationship between borrower demand and interest rate is given by $Q_t = g(\eta_t, \zeta_t)$ where Q_t is borrower demand and ζ_t is a random variable which represents environmental perturbations.

C. *Default Conditions*

We may model default of loans made with a random variable that represents percentage of failures. As a first approximation we ignore default.

D. *Payoffs and Dividends*

What does the bank maximize? It is easy to answer this question in a competitive market without uncertainty. There is one rate of interest and the bank maximizes the present value of the income stream. But when there is uncertainty and when the borrowing and lending rates are different, what rate is to be used for the bank's profits? Possibly we need a discount rate ψ ; the constraint $(1 + \rho)^{-1} \leq \psi \leq (1 + \eta)^{-1}$ can be defended on an ad hoc basis. For any finite horizon we need both the evaluation of an appropriately discounted expected dividend stream and a salvage value to the firm.

E. *Bank Failure*

If the bank cannot pay back its loan either we must specify failure conditions or we must consider the possibility of a rollover of debt. We assume that there is a lender of last resort who will lend at a price α_t adjusted to the circumstances and from which the bank must borrow if it cannot repay depositors from its cash reserves. This structure combined with the absence of default by borrowers avoids the issue of bank insolvency.

F. *Lending and Borrowing: Short and Long*

The minimal complexity required to investigate the problems of matching book is to divide either sources, or uses or both into long term and short term. In modeling long-term borrowing or lending several new features appear which must be made explicit. Is long-term financing done every period? The general and realistic answer is yes. Can a long-term loan be paid back in full before it is due? In practice the answer is often yes, but with a special clause calling for a penalty or even a bonus.

G. *Set Up Costs, Administration and Overheads*

An important feature in financial transactions in general is the cost of transactions. Items such as paperwork may not vary materially with loan size over broad ranges. Frequently refinancing introduces considerable cost. These nonconvex features are not dealt with directly in these models, although part of the reason for the difference between long-term and short-term rates is the difference in transactions costs.

H. *On Leverage*

In modeling the relationship between lending and a bank's asset structure, both law and custom have dictated the ratios between assets and liabilities. Thus there is a leverage ratio (m) between loans and deposits.

3. The Matched Book Bank

In this and the subsequent section we consider two basic models and note some variants. In particular the first model here is that of the individual bank which has a fully matched policy of borrowing and lending short term. In the model in §5 we consider the bank which borrows short term, but can lend long term or short term.

3.1. *The Model*

We assume that the bank pays out dividends which its owners consume and its objective function is to maximize the present value of the expected dividend stream.

Let a_t be the cash level at the start of period t at which time the borrowing and lending interest rates (ρ_t and η_t , respectively) are known. After the bank decides how much to borrow (D_t) and the amount of dividends to issue (p_t), the demand for loans (Q_t) is revealed. The constraints on borrowing and dividends are

$$0 \leq D_t \leq ma_t, \quad 0 \leq p_t, \quad p_t + \rho_t D_t \leq a_t, \quad (1)$$

where m is the leverage parameter. We assume that $(\rho_1, \eta_1, Q_1), (\rho_2, \eta_2, Q_2), \dots$ are independent and identically distributed triples; let (ρ, η, Q) have the distribution that is common to these triples.

We assume only in §3.3 that

$$P\{(1 + \rho)^{-1} < \psi < (1 + \eta)^{-1}\} = 1$$

where ψ is the bank's single-period discount factor. This "cheap money" assumption corresponds to

$$P\{\psi(1 + \eta) < 1 < \psi(1 + \rho)\} = 1. \quad (2a)$$

The event $\{\psi < (1 + \eta)^{-1}\}$ has the interpretation that the marginal revenue from loans is higher than the bank's hurdle rate for the profits of new short-term investments. The event $\{(1 + \rho)^{-1} < \psi\}$ has the interpretation that the bank's hurdle rate is higher than that of the source of funds from which D_t is borrowed. Corresponding to (2a), we assume that generic values of the parameters satisfy

$$\psi(1 + \eta) < 1 < \psi(1 + \rho) \quad (2b)$$

which implies $\eta < 1/\psi - 1$. We also assume $\rho < \rho^*$ for some $\rho^* < \infty$.

The dynamics are

$$a_{t+1} = y_t + \eta_t \cdot \min\{Q_t, y_t + D_t\} \quad \text{where} \quad (3a)$$

$$y_t = a_t - \rho_t D_t - p_t. \quad (3b)$$

In words, cash at the beginning of the next period equals cash at the beginning of this period minus interest paid minus dividends issued plus interest received (on the lesser of cash available and loans demanded). From (3a,b)

$$\begin{aligned} p_t &= a_t - y_t - \rho_t D_t \\ &= y_{t-1} - y_t - \rho_t D_t + \eta_{t-1} \cdot \min \{ Q_{t-1}, y_{t-1} + D_{t-1} \}. \end{aligned}$$

Let ψ be the bank's single-period discount factor. Then the present value B of its dividend stream is

$$\begin{aligned} B &= \sum_{t=1}^{\infty} \psi^{t-1} p_t \\ &= \sum_{t=1}^{\infty} \psi^{t-1} (a_t - y_t - \rho_t D_t) \\ &= \sum_{t=1}^{\infty} \psi^{t-1} [\psi \eta_t \cdot \min \{ Q_t, y_t + D_t \} - \rho_t D_t - (1 - \psi) y_t] + a_1. \end{aligned} \quad (4a)$$

Let $\psi^{t-1} u_t$ be the summand in (4a). Since a_1 is known, a policy maximizes $E(B)$ if and only if it maximizes

$$E(B) - a_1 = E \left(\sum_{t=1}^{\infty} \psi^{t-1} u_t \right).$$

For the remainder of this section, we analyze the simpler problem of maximizing

$$E \left(\sum_{t=1}^n \psi^{t-1} u_t \right) \quad (4b)$$

for $n < \infty$. However, we show in §6 that the qualitative results for criterion (4b) are valid for criterion (4a).

It follows from (1) and

$$u_t = \psi \eta_t \cdot \min \{ Q_t, y_t + D_t \} - \rho_t D_t - (1 - \psi) y_t \quad (4c)$$

that the maximization of (4b) corresponds to the following dynamic programming recursion:

$$v_n(a, \rho, \eta) = \max \{ J_n(y, D; \rho, \eta) : 0 \leq y \leq a, 0 \leq D \leq ma \}, \quad (5)$$

$$\begin{aligned} J_n(y, D; \rho, \eta) &= \psi \eta E[\min \{ Q, y + D \}] - \rho D - (1 - \psi) y \\ &\quad + \psi E[v_{n-1}(y + \eta \cdot \min \{ Q, y + D \}, \rho, \eta)] \end{aligned} \quad (6)$$

for $n = 1, 2, \dots$, where $v_0(\cdot, \cdot, \cdot) = 0$. We assume that the distribution of (ρ, η, Q) permits the expectations in (6) to exist finitely and that the maximum in (5) is always achieved.

In the following subsection and later in the paper we partially order sets T of pairs of real numbers (u_1, u_2) as follows. If $(u_1, u_2) \in T$ and $(u'_1, u'_2) \in T$ then $(u_1, u_2) \leq (u'_1, u'_2)$ if $u_1 \leq u'_1$ and $u_2 \leq u'_2$. We say that (u_1^*, u_2^*) is *maximal* if $(u_1^*, u_2^*) \in T$ and $(u_1^*, u_2^*) \geq (u_1, u_2)$ for all $(u_1, u_2) \in T$.

3.2. Solution Properties

An inductive proof establishes that $v_n(\cdot, \cdot, \cdot)$ and $J_n(\cdot, \cdot; \rho, \eta)$ are continuous concave functions on their respective domains for all $n, \eta \geq 0$, and $\rho \geq 0$.

Let $T_n(\rho, \eta)$ be the set of (y, D) that maximizes $J_n(\cdot, \cdot; \rho, \eta)$ on the nonnegative orthant. It follows from Topkis (1978) that (i) for each ρ and η , $T_n(\rho, \eta)$ has a maximal element $[y_n(\rho, \eta), d_n(\rho, \eta)]$ and (ii) $\rho \geq \rho'$ and $\eta \leq \eta'$ implies $[y_n(\rho, \eta), d_n(\rho, \eta)] \leq [y_n(\rho', \eta'), d_n(\rho', \eta')]$. Concavity implies $(y, D) = [y_n(\rho, \eta), d_n(\rho, \eta)]$ is optimal in (5) if $y_n(\rho, \eta) \leq a$ and $d_n(\rho, \eta) \leq ma$. If $a < y_n(\rho, \eta)$ and $d_n(\rho, \eta) \leq ma$, then $y = a$ is optimal; hence $p = D = 0$. If $a \geq y_n(\rho, \eta)$ and $d_n(\rho, \eta) > ma$ then $D = ma$ is optimal. If $y_n(\rho, \eta) > a$ and $d_n(\rho, \eta) > ma$ then it is optimal either for $y = a$ (hence $D = 0$) or $D = ma$ (or both).

In summary, an optimal policy specifies

$$\begin{aligned} p &= a - y_n(\rho, \eta) - \rho d_n(\rho, \eta) & \text{and} & & D &= d_n(\rho, \eta) \\ & & \text{if} & & y_n(\rho, \eta) &\leq a \quad \text{and} \quad d_n(\rho, \eta)/m \leq a, \\ p &= D = 0 & \text{if} & & d_n(\rho, \eta)/m &\leq a < y_n(\rho, \eta), \\ D &= ma & \text{if} & & y_n(\rho, \eta) &\leq a < d_n(\rho, \eta)/m. \end{aligned}$$

Suppose that $\mathbf{Q}$ and (ρ, η) are independent. This assumption implies that demand is totally inelastic with respect to "price" ρ . In §5, we discuss a more realistic model with elastic demand. Concavity, (5), and (6) imply that $y_n(\rho, \eta)$ and $d_n(\rho, \eta)$ are nonincreasing in ρ and nondecreasing in η . Therefore, (i) the dividend, p , is nonincreasing as the rate of interest at which the bank lends, η , increases, (ii) the amount it borrows, D , is nondecreasing as the unit revenue it receives, η , increases. The dividend, p , is *not* a monotonic function of ρ as the next section illustrates.

3.3. Last Period

When $n = 1$ in (5),

$$J_1(y, D; \rho, \eta) = \psi \eta E[\min \{\mathbf{Q}, y + D\}] - \rho D - (1 - \psi)y.$$

Let $\Phi(\cdot)$ be the (marginal) distribution function of $\mathbf{Q}$ and let $\Phi(0) = 0$.

We now show that cash reserves in the last period are split completely between dividend payment and interest payment for deposits. Also there are two cases which determine the amount of deposits the bank should accept. If $\psi\eta \leq \rho$ then $D = 0$; if $\psi\eta > \rho$ then $D = \Phi^{-1}[1 - \rho/\psi\eta]$ where $\Phi^{-1}(\alpha) = \sup \{x: \Phi(x) \leq \alpha\}$. Let $J_n^{(1)}$ and $J_n^{(2)}$ denote right-hand derivatives $\partial J_n / \partial y$ and $\partial J_n / \partial D$, respectively. Then

$$J_1^{(1)}(y, D; \rho, \eta) = -(1 - \psi) + \psi\eta[1 - \Phi(y + D)], \quad (7)$$

$$J_1^{(2)}(y, D; \rho, \eta) = -\rho + \psi\eta[1 - \Phi(y + D)]. \quad (8)$$

Since $\Phi(\cdot)$ is a distribution function,

$$J_1^{(1)}(y, D; \rho, \eta) \leq (1 - \psi) + \psi\eta[1 - \Phi(d)] \leq -(1 - \psi) + \psi\eta, \quad (9)$$

$$J_1^{(2)}(y, D; \rho, \eta) \leq -\rho + \psi\eta[1 - \Phi(y)] \leq -\rho + \psi\eta. \quad (10)$$

From (1b), $J_1^{(1)} \leq 0$ for all y ; so $y = 0$ (hence $a = p + \rho D$) is optimal. In (10), $y = 0$ and $\Phi(0) = 0$ imply $J_1^{(2)} \leq -\rho + \psi\eta$; so $D = 0$ is optimal if $\psi\eta \leq \rho$.

From (8) and $y = 0$, an interior optimum with respect to D satisfies

$$D = \Phi^{-1}[1 - \rho/(\psi\eta)]. \quad (11)$$

So $D = 0$ if $\psi\eta \leq \rho$.

In summary, in the last period, $y = 0$ is optimal, i.e.

$$p = a - \rho D \quad (12)$$

and (11) specifies D . So $\psi\eta \leq \rho$ implies $D = 0$ and $p = a$, i.e. all cash is distributed as a dividend, no funds are borrowed, and none are lent. If $\psi\eta > \rho$ then (11) specifies the amount borrowed and (12) specifies the dividend.

EXAMPLE. Suppose that $\mathbf{Q}$ has a marginal distribution which is exponential with $E(\mathbf{Q}) = 1/\mu$:

$$\Phi(x) = 1 - e^{-\mu x} \quad (x \geq 0, \mu > 0).$$

Then

$$\Phi^{-1}(\alpha) = -[\log(1 - \alpha)]/\mu \quad (0 \leq \alpha < 1).$$

So (11) is

$$D = -\{\log[(\rho/\psi\eta)]\}/\mu \quad (\rho < \psi\eta),$$

and the dividend (12) is

$$p = a + (\rho/\mu) \log[(\rho/\psi\eta)] \quad (\rho < \psi\eta).$$

Therefore, p increases with ρ if $1 < \psi\eta/\rho < e$ and decreases with ρ if $e \leq \psi\eta/\rho < \infty$.

If the bank can obtain funds at 6% and lend at 10%, and if its discount factor corresponds to 8%, then $\rho = 0.06$, $\eta = 0.01$, and $\psi = (1.08)^{-1}$. So (11) is

$$D = -E(\mathbf{Q}) \log[(\rho/\psi\eta)] = E(\mathbf{Q}) \log(0.648) = 0.188E(\mathbf{Q}).$$

That is, in the last period the bank borrows approximately 19% of the amount which it expects to be demanded. With a rectangular distribution on $\frac{1}{2}$ to $\frac{3}{2}$ and the other parameters as above (11) gives $D = 0.85E(\mathbf{Q})$.

4. Interest as a Function of Liquidity

Recall that liquidity risk is the possibility that depositors may elect to withdraw part or all of their funds. One of the ways to model this risk is to relate the bank's condition to the interest that the bank must pay to obtain funds. In this section, we let ρ_t be a function of the cash level (a_t), the loan size (D_t), or minimum balance (y_t). That is, for a function $g(\cdot)$ let ρ_t be given by $g(s_t)$, $g(D_t)$, or $g(y_t)$. Then in (4) the term $-\rho_t D_t$ is replaced by $-D_t g(a_t)$, $-D_t g(D_t)$, or $-D_t g(y_t)$. Similarly, the term $-\rho D$ in (6) is replaced by $-Dg(s)$, $-Dg(D)$, or $-Dg(y)$.

If $g(\cdot)$ is a nondecreasing convex function on $[0, \infty)$, then the concavity properties previously asserted for v_t and J_t remain valid. It follows that the previously stated monotonicity properties of an optimal selection of y and D are preserved. However, if $g(\cdot)$ is nonincreasing and convex as is more likely, the concavity properties may be in peril.

4.1. Last Period

Suppose $\rho_t = g(D_t)$ for each t where $g(\cdot)$ is a nondecreasing, convex, and differentiable function. Then the previous analysis which concludes that $y = 0$ in the last period is unaltered. So $y = 0$ is optimal here. Then (8) becomes

$$J_1^{(2)}(y, D; \eta) = \psi\eta[1 - \Phi(D)] - g(D) - Dg'(D).$$

So $D = 0$ is optimal if $g(0) \geq \psi\eta$. If there is an interior optimum,

$$\psi\eta = g(D) + Dg'(D) + \psi\eta\Phi(D). \quad (13)$$

Similarly, if $\rho_t = g(a_t)$ for each t and (1b) is valid when ρ is replaced by $g(0)$, then the same analysis is valid as when ρ_t is exogenous. That is, in the last period $y = 0$ is optimal and

$$D = \Phi^{-1}[1 - g(a)/(\psi\eta)].$$

If $\rho_t = g(y_t)$ for each t and (1b) is valid when ρ is replaced by $g(D)$, then an optimal selection is $y = 0$ and

$$D = \Phi^{-1}[1 - g(0)/(\psi\eta)].$$

For example, if $\Phi(x) = 1 - e^{-\mu x}$, then

$$D = -\{\log [g(0)/(\psi\eta)]\}/\mu$$

if $g(0) \leq \psi\eta$. Otherwise, $D = 0$ because the "prime rate," $g(0)$, is too high.

4.2. Loss Reserves

Suppose that ξ_t is the fraction of loan dollars on which borrowers in period t default. Then (3a) is altered to

$$a_{t+1} = y_t + (\eta_t - \xi_t) \min \{Q_t, y_t + D_t\}.$$

If $(\rho_1, \eta_1, \xi_1, Q_1), (\rho_2, \eta_2, \xi_2, Q_2), \dots$ are independent and identically distributed vectors such that

$$P\{\eta_1 \geq \xi_1\} = 1, \quad (14)$$

then the results thus far remain valid when η is replaced by $\eta - \xi$. If (14) is false, then the model is more difficult to analyze.

4.3. Description of Policy and Sensitivity Analysis

We have shown above that there is an optimal policy which is more or less consistent with our intuition. The decision variables are dividends and how much to borrow (or levels of deposits to be accepted). These variables will be influenced by changes in the basic parameters of the problem. In particular these are (i) the initial capital, (ii) the cost of money to the bank, (iii) the lending rate, (iv) the bank's profit discount factor and (v) changes in the probability distributions.

5. The Bank With Short-Term Borrowing and Long- and Short-Term Lending

In this model we consider the added complication introduced by a term structure in loans. This complication can enter on both the asset and liability sides. But as a simplification for the sake of more tractability we omit long-term borrowing by the bank. Thus we concentrate on how much capital the bank should have and how much long-term lending it might wish to assume given that it borrows short term.

The bank can obtain short-term money from three sources: (1) its own capital which we denote initially by a_1 , (2) short-term deposits $D_t = f(\rho_t, \zeta_t)$ [the level of deposits is a function of the interest offered by the bank and a random component ζ_t], and (3) the lender of last resort or the central bank which lends funds at the rate α_t which may be fixed in advance or be a function of the capital and size of debt of the bank.

The bank can lend money short or long. We shall, as a crude approximation, model this with one period for short and two periods for long. Let the annual rate for short be η_t and for long be β_t . Long-term rates can be larger or smaller than short-term rates depending on transactions costs and expectations.

The needs of bank clients are differentiated by their business requirements; thus there may be some substitutability between long-term and short-term borrowing but it is not necessarily high. We may represent the demand for short-term and long-term borrowing as $S_t = g_1(\eta_t, \beta_t, \zeta_t)$ and $L_t = g_2(\eta_t, \beta_t, \zeta_t)$ where $\zeta_1, \zeta_2, \dots$ are independent and identically distributed random variables.

The competitive bank, if small, will more or less have to accept the rates at which it lends and borrows, as dictated by outside forces. Thus its strategy is its policy with respect to capital structure, borrowing and lending.

We still need to specify a payoff function for the bank. This, as before, may be the expected present value of dividends (plus the salvage value) where the discount rate ψ has to be justified.

5.1. Model

Let S_t and L_t be the exogenous demands for short-term and long-term loans in period t , respectively. Repayments on S_t and L_t are due at the ends of periods t and $t + 1$, respectively. However, the bank may regulate the term structure of its loan portfolio by electing to lend less than S_t or L_t . Let x_t and z_t be the respective amounts of short-term and long-term loans made in period t at interest rates η_t and β_t . Of course,

$$0 \leq x_t \leq S_t, \quad 0 \leq z_t \leq L_t. \quad (15)$$

Credit rationing occurs if $x_t < S_t$ or $z_t < L_t$. Belloc and Freixas (1988) discuss the roles of adverse selection and moral hazard in credit markets as consequences of banks' decisions to grant or withhold credit.

We assume that S_t and L_t are revealed after the dividend p_t is announced. Then x_t and z_t are selected subject to (15) and a liquidity constraint we shall specify below. Suppose that the level of demand deposits and the interest rate on those deposits, D_t and ρ_t , respectively, are known when the dividend p_t is selected. Let d_t be the amount of the deposits which is accepted; so

$$0 \leq d_t \leq D_t \quad \text{and} \quad d_t \leq m a_t \quad (16)$$

where m is the leverage parameter. We assume that the assets must be sufficient to cover the dividends and the interest on deposits. Let a_t be the bank's liquid assets at the beginning of period t . So

$$0 \leq a_t - p_t - \rho_t d_t, \quad 0 \leq p_t.$$

Let

$$y_t = a_t - p_t - \rho_t d_t, \quad (17)$$

so

$$0 \leq y_t \leq a_t. \quad (18)$$

The bank has $y_t + d_t$ available to lend, but it may elect to lend a total, $x_t + z_t$, which exceeds the available funds. In this event, the excess must be obtained from a central bank of last resort at a (presumably) high interest rate α_t . We assume that α_t , η_t , and β_t are known when x_t and z_t are chosen.

Under these assumptions

$$a_{t+1} = y_t + \eta_t x_t - \alpha_t(x_t + z_t - d_t - y_t)^+ + (1 + \beta_{t-1})z_{t-1} - (1 - \beta_t)z_t. \quad (19)$$

The last two terms on the right side of (19) refer to repayment of the previous period's long-term loans with interest, and the long-term loans made in the current period.

We assume that the bank operates in a regulatory environment that mandates non-negative cash reserves. So it must select x_t and z_t in order to satisfy $a_{t+1} \geq 0$. From (19), this constraint corresponds to

$$\alpha_t(x_t + z_t - d_t - y_t)^+ - \eta_t x_t - (1 - \beta_t)z_t \leq y_t + (1 + \beta_{t-1})z_{t-1}. \quad (20)$$

We shall not analyze the model of a bank in a less restrictive regulatory environment in which cash reserves plus long-term loans must be nonnegative. In that case, the right side of (20) would be increased by z_t .

It follows from (17) and (19) that

$$\begin{aligned} p_t &= a_t - \rho_t d_t - y_t \\ &= -\rho_t d_t - y_t + a_{t-1} + \eta_{t-1} x_{t-1} - \alpha_{t-1} (x_{t-1} + z_{t-1} - d_{t-1} - y_{t-1})^+ \\ &\quad + (1 + \beta_{t-2}) z_{t-2} - (1 - \beta_{t-1}) z_{t-1}. \end{aligned} \quad (21)$$

The substitution of (21) and rearrangement of terms yields the present value

$$\begin{aligned} B = \sum_{t=1}^{\infty} \psi^{t-1} p_t &= a_1 + \sum_{t=1}^{\infty} \psi^{t-1} \{ -y_t - \rho_t d_t + \psi a_t + \psi \eta_t x_t \\ &\quad - \psi [1 - \psi - (1 + \psi) \beta_t] z_t - \psi \alpha_t (x_t + z_t - y_t - d_t)^+ \}. \end{aligned} \quad (22a)$$

As in §3, for the remainder of this section we analyze a finite-horizon version of (22a). Let $\psi^{t-1} u_t$ be the summand in (22a). Since a_1 is known, an optimal policy maximizes

$$E(B) - a_1 = E\left(\sum_{t=1}^{\infty} \psi^{t-1} u_t\right)$$

Thus, we analyze the problem of maximizing

$$E\left(\sum_{t=1}^n \psi^{t-1} u_t\right) \quad (22b)$$

and show in §6 that the qualitative results remain valid as $n \rightarrow \infty$.

Let $\lambda_t = (\alpha_t, \beta_t, \eta_t, S_t, L_t)$. In the stochastic process $(\rho_1, D_1, \lambda_1), (\rho_2, D_2, \lambda_2), \dots$, we assume that the distribution of (ρ_t, D_t) depends only on $(\beta_{t-2}, \lambda_{t-1})$ and the distribution of λ_t depends only on (β_{t-1}, ρ_t) . These assumptions encompass a wide variety of autocorrelated interest rate processes.

We use (15), (16), and (18) to formulate the maximization of (22b) as a dynamic program. Let λ be the information which is revealed after the dividend and deposit (p_t and d_t) are selected, namely $\lambda = (\alpha, \beta, \eta, S, L)$ and let β' and Z be the values of β and z in the previous period. It follows from (20) that $w = (1 + \beta')Z$ is an element of the constraint on x and z , given the other information. The state of the dynamic program is $[a, \rho, D, (1 + \beta')Z]$ and the decisions (y, d) are made at the beginning of the period. Then λ is revealed and (x, y) are selected. Let $v_0(\cdot) \equiv 0$ and for $n \geq 1$,

$$\begin{aligned} v_n(a, w, \rho, D) &= \psi a + \max \{ -y - \rho d + E[K_n(y, d, w, \lambda)]: \\ &\quad 0 \leq y \leq a, d \leq ma, 0 \leq d \leq D \}, \end{aligned} \quad (23a)$$

$$\begin{aligned} K_n(y, d, w, \lambda) &= \psi \max \{ J_n(x, z, y, d, w, \lambda): 0 \leq x \leq S, 0 \leq z \leq L, \\ &\quad \alpha(x + z - d - y)^+ - \eta x - (1 - \beta)z \leq y + w \}, \end{aligned} \quad (23b)$$

$$\begin{aligned} J_n(x, z, y, d, w, \lambda) &= \eta x - [1 - \psi - (1 + \psi)\beta]z - \alpha(x + z - y - d)^+ \\ &\quad + E\{v_{n-1}[y + \eta x - \alpha(x + z - y - d)^+ + w - (1 - \beta)z, (1 + \beta)z, \rho, \mathbf{D}]\}, \end{aligned} \quad (23c)$$

We assume that the maxima in (23a,b) are achieved and that expectations exist and are finite.

5.2. Solution Properties

An inductive proof that uses (23a) and $\psi \geq 0$ establishes that $v_n(\cdot, w, \rho, D)$ is a nondecreasing function for each (w, ρ, D) and that $v_n(\cdot)$, $K_n(\cdot, \lambda)$, and $J_n(\cdot, \lambda)$ are continuous concave functions on their respective domains (for each λ with nonnegative components).

Let $\xi = (\alpha, \beta, \eta, y, d, w)$, let $T_{n1}(w, \rho)$ be the set of (y, d) that maximize the objective in (23a) on the nonnegative orthant, and let $T_{n2}(\xi)$ be the set of (x, z) that maximize the objective in (23b) on the nonnegative orthant. It follows from Topkis (1978) that (i) $T_{n1}(w, \rho)$ has a maximal element $[y_n(w, \rho), d_n(w, \rho)]$ for each n, w , and ρ , and (ii) T_{n2} has a maximal element $[x_n(\xi), z_n(\xi)]$ for each n and ξ .

Concavity and (23a,b,c) lead to the following conclusions (in which the arguments of y_n, d_n, x_n , and z_n are suppressed) for optimal values of decision variables:

$$\begin{aligned}
 p &= a - y_n - \rho d_n \quad \text{and} \quad d = d_n \\
 &\quad \text{if} \quad y_n \leq a \quad \text{and} \quad d_n/m + (a - D/m)^+ \leq a, \quad (24) \\
 p &= d = 0 \quad \text{if} \quad d_n/m \leq a < y_n \quad \text{and} \quad d_n \leq D, \\
 d &= \min \{ma, D\} \quad \text{if} \quad y_n \leq a < d_n/m + (a - D/m)^+, \\
 x &= x_n \quad \text{and} \quad z = z_n \quad \text{if} \quad x_n \leq S, \quad z_n \leq L \quad \text{and} \\
 &\quad \alpha(x_n + z_n - d - y)^+ - \eta x_n - (1 - \beta)z_n \leq y + w, \\
 x &= S \quad \text{if} \quad x_n > S, \quad z_n \leq L, \quad \text{and} \\
 &\quad \alpha(x_n + z_n - d - y)^+ - \eta x_n - (1 - \beta)z_n > y + w, \quad (25) \\
 z &= L \quad \text{if} \quad x_n \leq S, \quad z_n > L \quad \text{and} \quad (25), \\
 &\quad (\alpha - \eta)x + (\alpha + \beta - 1)z = y + w + \alpha(d + y) \quad \text{if} \quad (25), \\
 &\quad (y + w)/\alpha \leq L, \quad \text{and} \quad (y + w)/(\alpha - \eta) \leq S \quad (\alpha > \eta).
 \end{aligned}$$

More generally, there is an optimal solution in (22b), $(x, z) = [X_n(S, L), Z_n(S, L)]$ (we have suppressed parameters except for S and L), such that for all $\gamma > 0$,

$$\begin{aligned}
 0 &\leq X_n(S, L + \gamma) - X_n(S, L) \leq \gamma, \\
 0 &\leq X_n(S + \gamma, L) - X_n(S, L) \leq \gamma, \\
 0 &\leq Z_n(S + \gamma, L) - Z_n(S, L) \leq \gamma, \\
 0 &\leq Z_n(S, L + \gamma) - Z_n(S, L) \leq \gamma. \quad (26)
 \end{aligned}$$

That is, the amount of loans made of each type is monotonic in its demand, but does not increase faster than the demand. Similarly, there is an optimal solution $(y, D) = [Y_n(a, D), \delta_n(a, D)]$ in (23a) such that

$$\begin{aligned}
 0 &\leq Y_n(a + \gamma, D) - Y_n(a, D) \leq \gamma, \\
 0 &\leq Y_n(a, D + \gamma) - Y_n(a, D) \leq \gamma, \\
 0 &\leq \delta_n(a + \gamma, D) - \delta_n(a, D) \leq \gamma, \\
 0 &\leq \delta_n(a, D + \gamma) - \delta_n(a, D) \leq \gamma \quad (27)
 \end{aligned}$$

for all $\gamma > 0$. The interpretation is similar to that of (26). Also, the accepted deposit δ_n is a nonincreasing function of the deposit interest rate ρ and Y_n is invariant with respect to ρ ; so the dividend must be a nondecreasing function of ρ . The short-term and long-term loans (X_n and Z_n) are nonincreasing functions of the interest rate for “last resort” funds (α), X_n is a nondecreasing function of η (the interest rate on short-term loans) and Z_n is a nondecreasing function of β (the interest rate on long-term loans).

6. Matching with an Infinite Horizon

The analyses in §3 and §5 replaced infinite horizon planning horizons in (4a) and (22a) with finite horizon planning horizons ($n < \infty$) in (4b) and (22b). In this section we show that the earlier conclusions regarding the qualitative properties of an optimal policy remain valid for (4a) and (22a). Our method draws on Heyman and Sobel (1984,

sections 8-5 and 8-6) and outlines a proof that (a) the value functions of the finite horizon dynamic programs converge as $n \rightarrow \infty$, (b) the limit value function satisfies the functional equation of dynamic programming, (c) as $n \rightarrow \infty$, the finite horizon optimal policies converge to a policy that is optimal in the functional equation, and (d) the limit policy inherits the qualitative properties of the finite horizon optimal policies.

Model in §3

First, we consider the model in §3 where each period the bank reviews its cash level (a_t) and decides how much to borrow (D_t) and to issue as dividends (p_t). The following argument shows that v_n and J_n , the dynamic programming value functions in (5) and (6), converge as $n \rightarrow \infty$.

Let $u = E(\eta Q)\psi/(1 - \psi)$. It follows from (1), (3a,b), and (4c) that v_n , the value function for the maximization of (4b), satisfies

$$v_n \leq E \sum_{t=1}^n \psi^{t-1} \psi \eta_t Q_t \leq u \quad (28)$$

and in (6) $J_n \leq \psi E(Q) + \psi u$ under the assumption $P\{\eta \leq 1\} = 1$. For lower bounds, we observe in (5) that $v_n(0, \cdot, \cdot) \equiv 0$ and $v_n(a, \rho, \eta) \geq 0$ because $y = D = 0$ is always feasible. Finally, the assumptions $P\{\rho \geq 1 - \psi\} = 1$ and $P\{Q \leq K\} = 1$ imply $v_n(a, \rho, \eta) \leq v_{n+1}(a, \rho, \eta)$ and $J_n(y, D; \rho, \eta) \leq J_{n+1}(y, D; \rho, \eta)$ for all n . Therefore, v_n and J_n are bounded monotone sequences and there exist

$$v(a, \rho, \eta) = \lim_{n \rightarrow \infty} v_n(a, \rho, \eta) \quad \text{and} \quad J(y, D; \rho, \eta) = \lim_{n \rightarrow \infty} J_n(y, D; \rho, \eta). \quad (29)$$

Continuity and concavity of $v_n(\cdot)$ and $J_n(\cdot, \cdot; \rho, \eta)$ on their respective domains for each $\rho \geq 0$, $\eta \geq 0$, and n and monotone convergence imply that $v(\cdot)$ and $J(\cdot, \cdot; \rho, \eta)$ inherit these properties. Since the set of feasible actions is compact at each state, it follows that

$$v(a, \rho, \eta) = \max \{J(y, D; \rho, \eta) : 0 \leq y \leq a, 0 \leq D \leq ma\}. \quad (30)$$

Let $T(\rho, \eta)$ be the set of (y, D) that maximize the objective in (30) on the nonnegative orthant. It follows from Topkis (1978) that (i) for each ρ and η , $T(\rho, \eta)$ has a maximal element $[y(\rho, \eta), d(\rho, \eta)]$ and (ii) $\rho \geq \rho'$ and $\eta \leq \eta'$ implies $[y(\rho, \eta), d(\rho, \eta)] \leq [y(\rho', \eta'), d(\rho', \eta')]$. If $y(\rho, \eta) \leq a$ and $d(\rho, \eta) \leq ma$ then $(y, D) = [y(\rho, \eta), d(\rho, \eta)]$ is optimal in (27). If $a < y(\rho, \eta)$ and $d(\rho, \eta) \leq ma$, then $y = a$ and $D = 0$ are optimal. If $a \geq y(\rho, \eta)$ and $d(\rho, \eta) > ma$ then $D = ma$ is optimal. If $y(\rho, \eta) > a$ and $d(\rho, \eta) > ma$ then it is optimal either for $y = a$ (hence $D = 0$) or $D = ma$ (or both).

In summary, an optimal policy specifies

$$\begin{aligned} p &= a - y_n(\rho, \eta) - \rho d_n(\rho, \eta) & \text{and} & & D &= d_n(\rho, \eta) \\ & & \text{if} & & y_n(\rho, \eta) &\leq a \quad \text{and} \quad d_n(\rho, \eta)/m \leq a, \\ p &= D = 0 & \text{if} & & d_n(\rho, \eta)/m &\leq a < y_n(\rho, \eta), \\ D &= a & \text{if} & & y_n(\rho, \eta) &\leq a < d_n(\rho, \eta)/m. \end{aligned}$$

Model in §5

We outline here how the same argument as in the previous subsection leads to the conclusion that the infinite horizon counterparts of (23a,b,c), (24), (26), and (27) are valid. A bound similar to (28) can be specified for (23a,b,c) which yields versions of (29) for v_n , K_n , and J_n with n suppressed. Hence, the counterpart of (30) is valid, namely (23a,b,c) with n suppressed. Also, $v(\cdot, w, \rho, D)$ is nondecreasing for each (w, ρ, D) and $v(\cdot)$, $K(\cdot, \lambda)$, and $J(\cdot, \lambda)$ are continuous concave functions on their respective domains. Therefore, the infinite horizon counterparts of (24), (26), and (27) are valid.

7. Further Considerations

The analysis presented above hopefully offers some insight into the problems of matching long-term and short-term portfolios. But this is a highly oversimplified view of the many features involved in understanding time structure in financing. Some of these factors are noted below.

7.1. *Partial Equilibrium versus General Equilibrium*

The models we have considered are open or partial equilibrium models. If one tries to close them, several key logical difficulties emerge. How do we explain differences in borrowing and lending rates beyond the existence of transactions costs? Is it possible for individuals to hold different expectations over time without convergence to consistent rational expectations?"

7.2. *On Bankruptcy*

It generally simplifies the mathematics to treat the problems of illiquidity and insolvency continuously by stipulating costlier financing when these events arise. In actuality, banks or firms are liquidated or reorganized. In the reorganization the assets of a manufacturing firm tend to be far less liquid and more difficult to evaluate than those of a financial firm. Furthermore, the partial equilibrium open model of bankruptcy does not adequately illustrate the efficiency or distributional effects in restructuring debt.

7.3. *Time Structure, Technology and Expectations*

In actuality the needs of corporations, other institutions and individuals require a debt structure that can run out for 20 or 30 years. The durability, immovability and special purpose of many buildings, machines and equipment causes the problem of matching book to be far harder for the nonfinancial firm than for the financial institution.

The need to match long and short structure is present even with consistent expectations, as an appropriate defense against random fluctuations. The bank's (or firm's) capital is a money inventory to protect against fluctuations. What constitutes an optimal level of capital can be answered for the bank or firm in the type of models noted above. But although we believe that such a question merits answering it casts only partial light on the questions of the relative merits of the carrot of the availability of a lender of last resort or the whip of bankruptcy if protection provided by one's own capital is inadequate.¹

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